$\overline{\partial}$-APPROACH TO THE DISPERSIONLESS KP HIERARCHY *

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Abstract

The dispersionless limit of the scalar nonlocal $\overline{\partial}$-problem is derived. It is given by a special class of nonlinear first-order equations. A quasi-classical version of the $\overline{\partial}$-dressing method is presented. It is shown that the algebraic formulation of dispersionless hierarchies can be expressed in terms of properties of Beltrami tupe equations. The universal Whitham hierarchy and, in particular, the dispersionless KP hierarchy turn out to be rings of symmetries for the quasi-classical $\overline{\partial}$-problem.

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1 Introduction

Dispersionless or quasiclassical limits of integrable systems have atracted a considerable interest during the last years (see e.g. [1]-[13] and references therein). Such type of equations and hierarchies arise as a result of processes of averaging over fast variables or as a formal quasiclassical limit $\hbar$(or ϵ) $\to 0$. Study of dispersionless hierarchies is of great relevance since they play an important role in the analysis of various problems in different fields of physics and mathematics as, for example, the quantum theory of topological fields and strings [14]-[16], some models of optical communications [17] or the theory of conformal maps in the complex plane [18, 19].

Dispersionless hierarchies have been described and analysed by different methods. In particular, the quasiclassical versions of the inverse scattering transform and Riemann-Hilbert problem method have been applied to the study of some $1+1$-dimensional integrable equations [23, 20, 11, 12, 13]. In contrast, similar study of the $2+1$-dimensional integrable equations and hierarchies is missing. Our goal is to fill this gap.

In the present paper we shall approach the dispersionless hierarchies from the $\overline{\partial}$-dressing method. This method, based on the linear nonlocal $\overline{\partial}$-problem, is a very efficient tool for constructing and solving usual integrable hierarchies (see e.g. [21, 22, 23]. We shall demonstrate that this approach provides us with a new and promising viewpoint of the dispersionless hierarchies. First we shall derive the dispersionless (or quasiclassical) version of the $\overline{\partial}$-problem for the dKP hierarchy. It turns to be given by nonlinear first-order equations of the type

$$\frac{\partial S}{\partial \bar{z}} = W\Big(z, \bar{z}, \frac{\partial S}{\partial z}\Big). \tag{1}$$

It turns out that this type of equations are well-known in the theory of complex analysis, in connection with quasi-conformal mappings.

We shall formulate the quasi-classical version of the $\overline{\partial}$-dressing method and derive the dKP hierarchy using these equations. Moreover, we shall show that symmetries of the quasiclassical $\overline{\partial}$-problem are determined by a linear Beltrami equation and that they form an infinite-dimensional ring structure which constitutes nothing but the dKP hierarchy. In a more general setting this ring of symmetries coincides with the universal Whitham hierarchy introduced in [8].

2 Dispersionless KP and universal Whitham hierarchies

For the sake of convenience we remind here some relevant formulas for the standard and dispersionless KP hierarchies. The standard KP hierarchy written in Lax form (see e.g. [7, 10])

$$\frac{\partial L}{\partial t_n} = [(L^n)_+, L], \quad n = 1, 2, \dots, \tag{2}$$

arises as the compatibility conditions for the system

$$\begin{aligned} L\psi &= z\psi, \\ \frac{\partial \psi}{\partial t_n} &= (L^n)_+\psi, \quad n = 1, 2, \dots, \end{aligned} \tag{3}$$

where L is a pseudo-differential operator

$$L = \partial + u_1(\boldsymbol{t})\partial^{-1} + u_2(\boldsymbol{t})\partial^{-2} + \cdots,$$

with

$$\partial := \frac{\partial}{\partial x}, \quad \boldsymbol{t} := (t_1 := x, t_2, \dots),$$

$(L^n)_+$ denotes the pure differential part of L^n and $\psi = \psi(z, \boldsymbol{t})$ is a KP wave function.

The dKP hierarchy is given in Lax form by

$$\frac{\partial \mathcal{L}}{\partial T_n} = \{(\mathcal{L}^n)_+, \mathcal{L}\}, \quad n = 1, 2, \dots, \tag{4}$$

where $\mathcal{L} = \mathcal{L}(p, \boldsymbol{T})$ denotes a function which admits an expansion

$$\mathcal{L} = p + \frac{U_1(\boldsymbol{T})}{p} + \frac{U_2(\boldsymbol{T})}{p^2} + \dots, \quad p \to \infty, \quad \boldsymbol{T} := (T_1 := X, T_2, \dots), \tag{5}$$

$(\mathcal{L}^n)_+$ is the polynomial part of $\mathcal{L}^n$ as a function of p, and $\{,\}$ stands for the Poisson bracket

$$\{f, g\} := \frac{\partial f}{\partial p}\frac{\partial g}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial g}{\partial p}.$$

This system of equations can be derived as a formal $\epsilon \to 0$ limit of (2) under the change of variables $T_n = \epsilon t_n,\ n \geq 1$ [1]-[10]. In particular, the KP wave function is assumed to behave as

$$\psi(z, \frac{\boldsymbol{T}}{\epsilon}) \sim \exp\Big(\frac{S(z,\boldsymbol{T})}{\epsilon} + O(\epsilon^0)\Big),$$

where S can be expanded as

$$S(z,\boldsymbol{T}) = \sum_{n\geq 1} z^n T_n + \sum_{n\geq 1} \frac{S_n(\boldsymbol{T})}{z^n}, \quad z \to \infty, . \tag{6}$$

Under these assumptions it is easy to see that (3) reduces to (5) and

$$\frac{\partial S}{\partial T_n} = \Omega_n(p, \boldsymbol{T}), \tag{7}$$

where

$$p := \frac{\partial S}{\partial X}, \tag{8}$$

and $\Omega(p,\boldsymbol{T}) := (\mathcal{L}^n)_+$, with $\mathcal{L}(p,\boldsymbol{T}) := z(p,\boldsymbol{T})$ being the function provided by solving for z in (8).

The system (7) represents a family of Hamilton-Jacobi equations for S. Moreover, it can be shown [10] that given a function S of the form (6) which satisfies (7), then the corresponding function $\mathcal{L}$ is a solution of the dKP hierarchy (4). The compatibility conditions for (7) are given by

$$\frac{\partial \Omega_m}{\partial T_m} - \frac{\partial \Omega_n}{\partial T_n} + \{\Omega_n, \Omega_m\} = 0, \tag{9}$$

and represent the Zakharov-Shabat formulation of the dKP hierarchy.

A general scheme for generating dispersionless hierarchies is the universal Whitham hierarchy [8]. Its starting point is a Zakharov-Shabat system

$$\frac{\partial \Omega_A}{\partial T_B} - \frac{\partial \Omega_B}{\partial T_A} + \{\Omega_A, \Omega_B\} = 0, \tag{10}$$

with $\Omega_A = \Omega_A(p,\boldsymbol{T})$ being given meromorphic functions of p depending on a set of parameters $\boldsymbol{T}$. This hierarchy includes as particular cases the dKP, Benney and the dispersionless version of the 2-dimensional Toda lattice.

3 Quasi-classical $\bar{\partial}$-problems

The standard KP hierarchy is associated with the following scalar non-local linear $\bar{\partial}$ equation (see e.g. [21]-[23]) for the KP wave function

$$\frac{\partial \chi(z,\bar{z},\boldsymbol{t})}{\partial \bar{z}} = \iint_G \mathrm{d}\, z' \,\mathrm{d}\, \bar{z}' \chi(z',\bar{z}',\boldsymbol{t}) \psi_0(z',\boldsymbol{t}) R_0(z',\bar{z}',z,\bar{z}) \psi_0^{-1}(z,\boldsymbol{t}), \tag{11}$$

where G is a given bounded domain of $\mathbb{C}$, $\psi_0(z,\boldsymbol{t}) = \exp(\sum_{n\geq 1} z^n t_n)$ and $R_0 = R_0(z',\bar{z}',z,\bar{z})$ an appropriate function ($\bar{\partial}$-data). It is assumed that the function χ has a canonical normalization

$$\chi(z,\bar{z},\boldsymbol{t}) = 1 + \frac{\chi_1(\boldsymbol{t})}{z} + \frac{\chi_2(\boldsymbol{t})}{z^2} + \dots\,, \quad z \to \infty.$$

The corresponding wave function of the standard KP hierarchy is then given by $\psi = \psi_0 \cdot \chi$.

From (11) it follows that in order to get a non singular dispersionless limit of the corresponding solution of the KP hierarchy, the function

$$\begin{gathered} u(\frac{\boldsymbol{T}}{\epsilon}) := \epsilon \frac{\partial}{\partial T_1} \iint_G \mathrm{d}\, z \,\mathrm{d}\, \bar{z} \iint_G \mathrm{d}\, z' \,\mathrm{d}\, \bar{z}' R_0(z',\bar{z'},z,\bar{z}) \\ \times \exp\Big(\frac{1}{\epsilon}(S_0(z',\boldsymbol{T}) - S_0(z,\boldsymbol{T}))\Big), \quad S_0(z,\boldsymbol{T}) := \sum_{n\geq 1} z^n T_n, \end{gathered} \tag{12}$$

should have a finite $\epsilon \to 0$ limit. This holds, in particular, for $\bar{\partial}$-data of the form

$$R_0(z',\bar{z'},z,\bar{z}) = \sum_{k\geq 0} (-1)^k r_k(z,\bar{z}') \epsilon^{k-1} \delta^{(k)}(z' - z - \epsilon \alpha_k(z,\bar{z})), \tag{13}$$

where r_k and α_k are arbitrary functions and

$$\delta^{(k)}(z,\bar{z}) := \frac{\partial^k \delta(z,\bar{z})}{\partial z^k}.$$

If we now rewrite (11) as

$$\frac{\partial \ln \chi(z,\bar{z},\frac{\boldsymbol{T}}{\epsilon})}{\partial \bar{z}} = \iint_G \mathrm{d}\, z' \,\mathrm{d}\, \bar{z}' \psi(z',\bar{z'},\frac{\boldsymbol{T}}{\epsilon}) R_0(z',\bar{z'},z,\bar{z}) \psi(z,\bar{z},\frac{\boldsymbol{T}}{\epsilon})^{-1}, \tag{14}$$

and insert a kernel of the form (14), the limit $\epsilon \to 0$ leads to

$$\frac{\partial S}{\partial \bar{z}} = W(z, \bar{z}, \frac{\partial S}{\partial z}), \tag{15}$$

where

$$W(z, \bar{z}, \frac{\partial S}{\partial z}) := \sum_{k \geq 0} r_k(z, \bar{z}) \exp\Big(\alpha_k(z, \bar{z}) \frac{\partial S}{\partial z}\Big) \Big(\frac{\partial S}{\partial z}\Big)^k. \tag{16}$$

The above discussion suggests to take equation (15), for appropriate functions W, as the quasi-classical version of the linear $\bar{\partial}$-problem (11). The function S is widely used in the discussions of the dispersionless limit of the integrable hierarchies [4]-[10]. Within the $\bar{\partial}$-approach it is a non-holomorphic function of the spectral parameter and obeys the nonlinear $\bar{\partial}$-equation (15).

The nonlinear Beltrami type equation (15) is well-known in complex analysis. Under certain conditions on W it belongs to the class of nonlinear elliptic systems on the plane in the sense of Lavrent'ev [24, **?**, **?**]. On the other hand, solutions of equations of this type determine quasi-conformal maps of their domains of definition on the complex plane (see e.g. [24, 25]). The connection between dispersionless hierarchies and the theory of quasiconformal maps is an interesting problem which will be considered elsewhere.

4 Quasi-classical $\bar{\partial}$-dressing method

Now we will use the $\bar{\partial}$-problem (15) to formulate the dKP hierarchy. In what follows we will have in mind expressions for W in which the dependence on z and $\bar{z}$ are provided for compact supported functions so as to allow for solutions of (15) with asymptotic form (6).

Suppose given a solution $S = S(z, \bar{z}, \boldsymbol{T})$ of (15) which as $z \to \infty$ is of the form (5). Then

$$\frac{\partial}{\partial \bar{z}}\Big(\frac{\partial S}{\partial T_n}\Big) = W'\Big(z, \bar{z}, \frac{\partial S}{\partial z}\Big) \frac{\partial}{\partial z}\Big(\frac{\partial S}{\partial T_n}\Big), \quad n \geq 1, \tag{17}$$

where

$$W'(z, \bar{z}, \lambda) := \frac{\partial W(z, \bar{z}, \lambda)}{\partial \lambda}. \tag{18}$$

This means that all time derivatives of S satisfy the same family (dependent on the infinite set of parameters $\boldsymbol{T}$) of linear Beltrami equations

$$\frac{\partial\Phi}{\partial\bar{z}} = Q(z,\bar{z},\boldsymbol{T})\frac{\partial\Phi}{\partial z}, \tag{19}$$

where

$$Q(z,\bar{z},\boldsymbol{T}) := W'\Big(z,\bar{z},\frac{\partial S}{\partial z}\Big).$$

Together with $\frac{\partial S}{\partial T_n}$, any combination

$$\sum_k U_{n_k}(\boldsymbol{T})\Big(\frac{\partial S}{\partial T_{n_k}}\Big)^{m_k},$$

obeys (17) as well. On the other hand, under mild conditions [25, 26], a solution of the linear Beltrami equation bounded on the whole plane $\mathbb{C}$ and vanishing at $z\to\infty$ vanishes identically. These properties are fundamental for our formulation of the dKP hierarchy. Indeed, given a solution of (15) of the form (5) then it follows that

$$\frac{\partial S}{\partial T_n} = z^n + \sum_{m\geq 1}\frac{S_m(\boldsymbol{T})}{z^m}, \quad z\to\infty, \tag{20}$$

and, in particular, the function $p := \frac{\partial S}{\partial X}$ can be expanded as

$$p = z + \sum_{n\geq 1}\frac{\partial_X S_n}{z^n}, \quad z\to\infty. \tag{21}$$

In this way, if we denote by $\mathcal{L}(p,\boldsymbol{T})$ the expansion for z obtained by inverting (21), it is clear from (28) and (21) that

$$\frac{\partial S}{\partial T_n} - (\mathcal{L}^n)_+ = O(\frac{1}{z}), \quad z\to\infty.$$

Hence, as

$$\frac{\partial S}{\partial T_n} - (\mathcal{L}^n)_+$$

is also a solution of (19) we conclude that S satisfies (7), so that $\mathcal{L}(p,\boldsymbol{T})$ is a solution of the dKP hierarchy. Therefore, we see that the quasi-classical $\bar{\partial}$-problem (15) leads to a straightforward formulation of the dKP hierarchy along the standard logic of the $\bar{\partial}$-dressing method.

5 Ring of symmetries of the quasi-classical $\bar{\partial}$-problem and the universal Whitham hierarchy

In section 3 we have derived the $\bar{\partial}$-problem (15) by starting with the $\bar{\partial}$-problem (11) for the KP hierarchy. One can show that the quasi-classical cs of the form (15) arise also as the dispersionless limit of other scalar integrable hierarchies, like the 2-dimensional Toda lattice and the modified KP hierarchy. The only difference consists in the different behaviours assumed for S at infinity.

Thus the problem (15), taken on some bounded domain G of $\mathbb{C}$, can be regarded as the starting point of a whole approach to scalar dispersionless hierarchies without any reference to the linear $\bar{\partial}$-problems of the original *dispersionfull* hierarchies. The main feature of this approach is that the symmetries (first order variations δS) of (15) are determined by the Beltrami equation

$$\frac{\partial}{\partial \bar{z}}(\delta S) = W'\Big(z, \bar{z}, \frac{\partial S}{\partial z}\Big)\frac{\partial}{\partial z}(\delta S). \tag{22}$$

As a consequence, if δS is a solution of (22) and $\Phi(\xi)$ is a differentiable function, then $\Phi(\delta S)$ is a solution too. Reciprocally, given two solutions $\delta_i S$ $(i = 1, 2)$ there exists a function $\Phi(\xi)$ such that $\delta_1 S = \Phi(\delta_2 S)$. Moreover, the product of two symmetries $\delta_1 S \delta_2 S$ also satisfies (22).

Therefore, symmetries of the quasi-classical $\bar{\partial}$-problem (15) form an infinite-dimensional ring.

Let us denote by T_A the times associated to the corresponding symmetry flows, so that the infinitesimal symmetries are $\frac{\partial S}{\partial T_A}$. If we mark one of such symmetries $\frac{\partial S}{\partial T_{A_0}}$ and denote it by p, then for any symmetry we can write

$$\frac{\partial S}{\partial T_A} = \Omega_A(p, \boldsymbol{T}), \tag{23}$$

for a certain function Ω_A. Thus, by varying Ω_A one can generate all the symmetries of the $\bar{\partial}$-problem (15) out of one of them p (basically arbitrary). The compatibility conditions for (23) are

$$\frac{\partial \Omega_A}{\partial T_B} - \frac{\partial \Omega_B}{\partial T_A} + \{\Omega_A, \Omega_B\} = 0, \tag{24}$$

where

$$\{f, g\} := \frac{\partial f}{\partial p}\frac{\partial g}{\partial A_0} - \frac{\partial f}{\partial A_0}\frac{\partial g}{\partial p}.$$

The infinite system (24) is exactly the Universal Whitham hierarchy. Hence, this hierachy describes the infinite-dimensional ring of symmetries of the scalar quasi-classical $\bar{\partial}$-problem (15). As it was shown in [8] the Universal Whitham hierarchy contains several relevant dispersionless hierarchies as particular reductions. This means that the quasi-classical $\bar{\partial}$-problem (15) has also a universal character.

6 Solutions of dispersionless hierarchies

Quasi-classical $\bar{\partial}$-dressing methods based on (15) provide us also with a tool for solving dispersionless hierarchies. The point is that we can apply the method of characteristics to solve (15) and then to find solutions S satisfying the appropriate behaviour at ∞.

To illustrate this process let us consider the dKP hierarchy and a $\bar{\partial}$-problem of the form

$$\frac{\partial S}{\partial \bar{z}} = \theta(1 - z\bar{z}) W_0\Big(\frac{\partial S}{\partial z}\Big), \tag{25}$$

where $\theta(\xi)$ is the usual Heaviside function and $W_0(m)$ is an arbitrary differentiable function. Observe that (25) implies

$$\frac{\partial m}{\partial \bar{z}} = W_0'(m)\frac{\partial m}{\partial z}, \quad m := \frac{\partial S}{\partial z}, \quad |z| < 1,$$

where $W_0' = \frac{\mathrm{d}\, W_0}{\mathrm{d}\, m}$. This equation can be solved at once by applying the methods of characteristics, so that the general solution S_{in} of (25) inside the unit circle $|z| < 1$ is implicitly characterized by

$$\begin{aligned} S_{in} &= W_0(m)\bar{z} + mz - f(m), \\ W_0'(m)\bar{z} + z &= f'(m), \end{aligned} \tag{26}$$

where $f = f(m)$ is an arbitrary differentiable function. The solution S_{out} of (25) outside the unit circle is any arbitrary analytic function ($\bar{z}$-independent).

However, in order to obtain a global solution of (25) in the class of locally integrable generalized functions we impose the continuity of S at the unit circle, so that

$$S_{out}(z) = S_{in}(z, \frac{1}{z}), \quad |z| = 1. \tag{27}$$

Moreover, as we are dealing with the dKP hierarchy, we require S_{out} to be of the form

$$S_{out} = \sum_{n\geq 1} z^n T_n + \sum_{n\geq 1} \frac{s_n(\boldsymbol{T})}{z^n}. \tag{28}$$

On the other hand, from (26) it follows that the coefficients of S_{out} are determined by the identity

$$\frac{\partial S_{out}}{\partial z} = m_0 - \frac{W(m_0)}{z^2}, \quad m_0(z) := m(z, \frac{1}{z}), \tag{29}$$

where m_0 is related to the arbitrary function f in (26) as

$$\frac{W'(m_0)}{z} + z = f'(m_0). \tag{30}$$

In this way we have a solution method for the dKP hierarchy based on solving the $\bar{\partial}$-equation (25). In principle the process is the following. We first take a certain function $f = f(m, \boldsymbol{a})$ depending on m and a certain set of undetermined parameters $\boldsymbol{a} = (a_1, \dots, a_n)$, and solve for $m = m(z, \bar{z}, \boldsymbol{a})$ in the second equation of (26). Then the functions $S_{in}(z, \bar{z}, \boldsymbol{a})$ and $S_{out}(z, \boldsymbol{a})$ are determined by means of the first equation of (26) and (27), respectively. Finally, we impose S_{out} to admit an asymptotic expansion (28) and get the parameters $\boldsymbol{a}$ as functions of the dKP times $\boldsymbol{t}$. However, as this method of solution is based on solving implicit equations, there is no general guarantee that the resulting expression for S be a global generalized solution of (25), so that a further analysis of the regularity of S is required. Furthermore, in case we were able to guarantee the appropriate regularity of S there would be an easier way to determine S. Indeed, we may start with a function $m_0 = m_0(z, \boldsymbol{a})$ and find the coefficients of the expansion (28) of S_{out} by imposing (28) and (30). According to (30), the corresponding function $f = f(m, \boldsymbol{a})$ in (26) is then given by

$$f'(m_0, \boldsymbol{a}) = \frac{W'(m_0)}{h(m_0, \boldsymbol{a})} + h(m_0, \boldsymbol{a}),$$

where $h(m_0, \boldsymbol{a})$ is the inverse function $z = h(m_0, \boldsymbol{a})$ of the function $m_0 = m_0(z, \boldsymbol{a})$.

As an example, let us consider the case

$$W_0(m) = m^2, \quad m_0 := az^2 + bz + c + \frac{d}{z}.$$

From (30) we get a function S_{out} of the form

$$S_{out} = \sum_{n=1}^{3} z^n T_n + \sum_{n=1}^{3} \frac{S_n(\boldsymbol{T})}{z^n},$$

where

$$a - a^2 = 3T_3, \quad (1-2a)b = 2T_2,$$

$$(1-2a)c - b^2 = X, \quad (1-2a)d - 2bc = 0,$$

$$S_1 = c^2 + 2bd, \quad S_2 = cd, \quad S_3 = \frac{d^2}{3}.$$

We notice that the function

$$U := -\frac{\partial S_1}{\partial X} = -\frac{2X + 24T_2^2 - 24T_3 X}{(1-12T_3)^2},$$

verifies the first equation of the dKP hierarchy

$$\frac{3}{2}\frac{\partial^2 U}{\partial T_2^2} = \frac{\partial}{\partial X}\Big(\frac{\partial U}{\partial T_3} - 6U\frac{\partial U}{\partial X}\Big).$$

A general discussion of the $\bar{\partial}$-method of solution for dispersionless hierarchies will be presented elsewhere.

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